Lesson 7-4

Example 1

If $\sin \theta = \frac{2}{5}$ and θ has its terminal side in the first quadrant, find the exact value of each function.

a. $\sin 2\theta$

To use the double-angle identity for $\sin 2\theta$, we must first find $\cos \theta$.

$$\sin^2 \theta + \cos^2 \theta = 1$$
$$\left(\frac{2}{5}\right)^2 + \cos^2 \theta = 1 \qquad \textit{sin } \theta = \frac{2}{5}$$
$$\cos^2 \theta = \frac{21}{25}$$
$$\cos \theta = \frac{\sqrt{21}}{5}$$

Then find $\sin 2\theta$.

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$= 21\frac{2}{5}21\frac{\sqrt{21}}{5}2 \qquad \textit{sin } \theta = \frac{2}{5}, \textit{ cos } \theta = \frac{\sqrt{21}}{5}$$
$$= \frac{4\sqrt{21}}{25}$$

b. $\cos 2\theta$

Since we know the values of $\cos \theta$ and $\sin \theta$, we can use any of the double-angle identities for cosine.

$$\cos 2\theta = 2\cos^2 \theta - 1$$
$$= 2\left(\frac{\sqrt{21}}{5}\right)^2 - 1 \qquad \textit{cos } \theta = \frac{\sqrt{21}}{5}$$
$$= \frac{17}{25}$$

c. tan 2θ

We must find tan θ to use the double-angle identity for tan 2θ.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{2}{5}}{\frac{\sqrt{21}}{5}} \qquad \textit{sin } \theta = \frac{2}{5}, \textit{ cos } \theta = \frac{\sqrt{21}}{5}$$

$$= \frac{2}{\sqrt{21}} \text{ or } \frac{2\sqrt{21}}{21}$$

Then find tan 2θ.

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2\left(\frac{2\sqrt{21}}{21}\right)}{1-\left(\frac{2\sqrt{21}}{21}\right)^2} \qquad \textit{tan } \theta = \frac{2\sqrt{21}}{21}$$

$$= \frac{\frac{4\sqrt{21}}{21}}{\frac{17}{21}} \text{ or } \frac{4\sqrt{21}}{17}$$

d. sin 4θ

Since 4θ = 2(2θ), use a double-angle identity for sine again.

$$\sin 4\theta = \sin 2(2\theta)$$

$$= 2\sin(2\theta)\cos(2\theta) \qquad \textit{Double-angle identity}$$

$$= 21\frac{4\sqrt{21}}{25}21\frac{17}{25}2 \qquad \textit{sin } (2\theta) = \frac{4\sqrt{21}}{25}, \textit{ cos } (2\theta) = \frac{17}{25}$$

$$= \frac{136\sqrt{21}}{625}$$

Example 2

Use a half-angle identity to find the exact value of each function.

a. $\tan \frac{5\pi}{8}$

$\tan \frac{5\pi}{8} = \tan \frac{\frac{5\pi}{4}}{2}$ *Use* $\tan \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$.

$= -\sqrt{\frac{1 - \cos \frac{5\pi}{4}}{1 + \cos \frac{5\pi}{4}}}$ *Since* $\frac{5\pi}{8}$ *is in Quadrant II, use the negative tangent value.*

$= -\sqrt{\frac{1 - \left(-\frac{\sqrt{2}}{2}\right)}{1 + \left(-\frac{\sqrt{2}}{2}\right)}}$

$= -\sqrt{\frac{2 + \sqrt{2}}{2 - \sqrt{2}}}$ or $-1 - \sqrt{2}$

b. $\sin 105°$

$\sin 105° = \sin \frac{210°}{2}$

$= \sqrt{\frac{1 - \cos 210°}{2}}$ *Use* $\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$.

$= \sqrt{\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{2}}$ *Since 105° is in Quadrant II, use the positive sine value.*

$= \frac{\sqrt{2 + \sqrt{3}}}{2}$

Example 3

Refer to the application at the beginning of the Lesson 7-4 in your book.

a. The water's straight line distance from its initial position to its maximum height is $\boldsymbol{S = \sqrt{\left(\frac{v^2}{g}\sin 2\theta\right)^2 + \left(\frac{v^2}{2g}\sin^2\theta\right)^2}}$ **. Simplify** $\boldsymbol{S}$**.**

b. If $\theta = 30°$, $v = 50$ **m/s, and** $g = 9.8\text{m/s}^2$**, what is the water's straight line distance from its initial position when it is at its maximum height?**

a.

$$S = \sqrt{\left(\frac{v^2}{g}\sin 2\theta\right)^2 + \left(\frac{v^2}{2g}\sin^2\theta\right)^2}$$

$$= \sqrt{\frac{v^4}{g^2}\sin^2 2\theta + \frac{v^4}{4g^2}\sin^4\theta}$$

$$= \sqrt{\frac{v^4}{g^2}(2\sin\theta\cos\theta)^2 + \frac{v^4}{4g^2}\sin^4\theta}$$ *Double-angle identity for sine*

$$= \sqrt{\frac{4v^4}{g^2}\sin^2\theta\cos^2\theta + \frac{v^4}{4g^2}\sin^4\theta}$$

$$= \sqrt{\frac{v^4}{g^2}\sin^2\theta 14\cos^2\theta + \frac{1}{4}\sin^2\theta 2}$$

$$= \frac{v^2}{g}\sin\theta\sqrt{4\cos^2\theta + \frac{1}{4}\sin^2\theta}$$

$$= \frac{v^2}{g}\sin\theta\sqrt{4\cos^2\theta + \frac{1}{4}(1 - \cos^2\theta)}$$ *Pythagorean identity:* $\sin^2\theta = 1 - \cos^2\theta$

$$= \frac{v^2}{g}\sin\theta\sqrt{4\cos^2\theta + \frac{1}{4} - \frac{1}{4}\cos^2\theta}$$

$$= \frac{v^2}{g}\sin\theta\sqrt{\frac{15}{4}\cos^2\theta + \frac{1}{4})}$$

$$= \frac{v^2}{2g}\sin\theta\sqrt{15\cos^2\theta + 1}$$

b. When $\theta = 30°$, $v = 50$ m/s, and $g = 9.8$ m/s^2, $S = \frac{50^2}{2(9.8)}\sin 30°\sqrt{15\cos^2 30° + 1}$, or about 223.21 meters from its initial position.

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Example 4

Verify that $\frac{-1 - \cos 2\theta}{\sin 2\theta} = -\cot \theta$ is an identity.

$$\frac{-1 - \cos 2\theta}{\sin 2\theta} \stackrel{?}{=} -\cot \theta$$

$$\frac{-1 - (2\cos^2 \theta - 1)}{2 \sin \theta \cos \theta} \stackrel{?}{=} -\cot \theta$$ *Double-angle identities for cosine and sine*

$$\frac{-2\cos^2 \theta}{2 \sin \theta \cos \theta} \stackrel{?}{=} -\cot \theta$$

$$-\frac{\cos \theta}{\sin \theta} \stackrel{?}{=} -\cot \theta$$

$$-\cot \theta = -\cot \theta$$